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On the Equivalence of Boundary Conditions

Jean Bricmont,[1] Joel L. Lebowitz,[1] and Charles E. Pfister[1]

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We show that if b and b' are two boundary conditions (b.c.) for general spin systems on $\mathbb{Z}^d$ such that the difference in the energies of a spin configuration σ_Λ in $\Lambda \subset \mathbb{Z}^d$ is uniformly bounded, $|H_{\Lambda,b}(\sigma_\Lambda) - H_{\Lambda,b'}(\sigma_\Lambda)| \leqslant C < \infty$, then any infinite-volume Gibbs states ρ and ρ' obtained with these b.c. have the same measure-zero sets. This implies that the decompositions of ρ and ρ' into extremal Gibbs states are equivalent (mutually absolutely continuous). In particular, if ρ is extremal, $\rho = \rho'$. Application of this observation yields in an easy way (among other things) (a) the uniqueness of the Gibbs states for one-dimensional systems with forces that are not too long-range; (b) the fact that various b.c. that are natural candidates for producing non-translation-invariant Gibbs states cannot lead to such an extremal Gibbs state in two dimensions.

KEY WORDS: Boundary conditions; Gibbs states; spin systems.

1. INTRODUCTION

The outline of this paper is as follows. We first recall briefly the definition and some known properties of Gibbs states for lattice systems. We then prove (by a simple observation) our main theorem and give various applications as corollaries. This is followed by some remarks about "stability" criteria for extremal Gibbs states and a sketch of a proof for extending our results to more general systems, e.g., lattice models with hard cores.

Gibbs States

We consider the general formalism of Gibbs states for spin systems with a compact metric space as phase space. We recall here the basic definitions and properties of Gibbs states. For more details, see Refs. 1 and 2.

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[1] Department of Mathematics, Rutgers University, New Brunswick, New Jersey.

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For each $i = (i_1, \dots, i_d) \in \mathbb{Z}^d$, we have a copy (Ω_i, ν_i) of a compact metric space Ω_0 and a probability measure ν_0 on Ω_0.

For $\Lambda \subseteq \mathbb{Z}^d$,

$$\Omega_\Lambda = \prod_{i\in\Lambda} \Omega_i, \qquad \nu_\Lambda = \prod_{i\in\Lambda} \nu_i$$

$\Omega = \Omega_{\mathbb{Z}^d}$ is equipped with the σ-algebra generated by the cylinder sets. For $\sigma \in \Omega$, $\sigma_\Lambda \in \Omega_\Lambda$ is the restriction of σ to Ω_Λ.

The group $\mathbb{Z}^d$ acts on Ω as

$$(T^i\sigma)_j = \sigma_{j-i}, \qquad \sigma \in \Omega \tag{1}$$

An interaction Φ is a family (Φ_X) indexed by the finite subsets of $\mathbb{Z}^d$ such that Φ_X is a continuous function on Ω_X that we identify with a function on Ω depending only on Ω_X,

$$\Phi_{X+i}(T^i\sigma) = \Phi_X(\sigma), \qquad \sum_{0\in X} \|\Phi_X\| < \infty \tag{2}$$

where $\|\cdots\|$ is the sup norm.

For example, in the usual Ising model with ferromagnetic nearest neighbor (n.n.) interactions, $\Omega_0 = \{-1, +1\}$, ν_0 gives weight 1/2 to -1 and $+1$, and

$$\begin{aligned} \Phi_X(\sigma_X) &= -J\sigma_i\sigma_j \qquad &&\text{for} \quad X = \{i, j\} \text{ n.n. sites} \\ &= 0 &&\text{otherwise} \end{aligned} \tag{3}$$

For a configuration $\sigma' \in \Omega$, one defines the *Hamiltonian* in the finite region $\Lambda \subset \mathbb{Z}^d$ under the *boundary condition* (b.c.) corresponding to specifying the spins outside Λ, $\sigma_{\bar\Lambda}'$ ($\bar\Lambda = \mathbb{Z}^d \backslash \Lambda$), as

$$H_{\Lambda,\sigma'}(\sigma) = \sum_{X\cap\Lambda\neq\varnothing} \Phi_X(\sigma_\Lambda, \sigma_{\bar\Lambda}')$$

and the corresponding *Gibbs measure in* Λ at reciprocal temperature β is then

$$\begin{aligned} \mu_{\Lambda,\sigma'}(d\sigma_\Lambda) &= Z_{\Lambda,\sigma'}^{-1} \exp(-\beta H_{\Lambda,\sigma'})\, \nu_\Lambda(d\sigma_\Lambda) \\ Z_{\Lambda,\sigma'} &= \int_{\Omega_\Lambda} \exp(-\beta H_{\Lambda,\sigma'})\, \nu_\Lambda(d\sigma_\Lambda) \end{aligned} \tag{4}$$

A *Gibbs state for* Φ is a measure ρ on Ω such that for any finite $\Lambda \subset \mathbb{Z}^d$ the restriction ρ_Λ of ρ to Ω_Λ satisfies

$$\rho_\Lambda = \int_{\Omega_{\bar\Lambda}} \mu_{\Lambda,\sigma'}\, \rho_{\bar\Lambda}(d\sigma_{\bar\Lambda}') \tag{5}$$

For $i \in \mathbb{Z}^d$, we define the translate of ρ, $T^i\rho$, by

$$(T^i\rho)(A) = \rho(T^{-i}A) \qquad \text{for} \quad A \subset \Omega \tag{6}$$

with $T^{-i}A$ defined by (1).

ρ is *translation invariant* if $T^i\rho = \rho$, $\forall i \in \mathbb{Z}^d$.

Before stating some properties of Gibbs states, we introduce *generalized b.c.* defined through the Hamiltonian:

$$H_{\Lambda,b}(\sigma_\Lambda) = \sum_{X\subset\Lambda} \Phi_X(\sigma_X) + \sum_{X\subset\Lambda} b_X{}^\Lambda(\sigma_X) \tag{7}$$

where $b_X{}^\Lambda$ is a continuous function on Ω_X and for any finite Λ

$$\lim_{\Lambda'\nearrow\mathbb{Z}^d} \left\| \sum_{X\cap\Lambda\neq\varnothing} b_X^{\Lambda'} \right\| = 0$$

$Z_{\Lambda,b}$ and $\mu_{\Lambda,b}$ are defined as in (4). One familiar example of generalized b.c. are the free b.c. corresponding to $b_X{}^\Lambda = 0$. Any b.c. defines a generalized b.c., due to (2), with

$$b_X{}^\Lambda = \sum_{\substack{X'\cap\bar\Lambda\neq\varnothing\\ X'\cap\Lambda = X}} \Phi_{X'} \tag{8}$$

Let G_Φ be the set of Gibbs states for Φ and $\overline{G_\Phi}$ the set of extreme points of G_Φ. The following is known:

(i) For any generalized b.c. b and any sequence $\Lambda_n \nearrow \mathbb{Z}^d$ such that $\lim_{n\to\infty} \mu_{\Lambda_n,b} = \rho$ exists (in the weak* sense), $\rho \in G_\Phi$.[2]

(ii) Moreover, for any $\rho \in G_\Phi$ and any sequence $\Lambda_n \nearrow \mathbb{Z}^d$, $\lim_{n\to\infty} \mu_{\Lambda_n,\sigma} \in \overline{G}_\Phi$ ρ a.e.[3]

(iii) Let B_Λ, $\Lambda \subset \mathbb{Z}^d$, be the σ-algebra generated by the cylinder sets based on Ω_Λ and $B_\infty = \bigcap_\Lambda B_{\bar\Lambda}$, where the intersection runs over the finite subsets of $\mathbb{Z}^d$. Then $\rho \in \overline{G}_\Phi$ if and only if $\rho(A) = 0$ or 1 for all $A \in B_\infty$.[1,2]

(iv) For any $\rho \in G_\Phi$, there exists a unique probability measure μ_ρ, concentrated on $\overline{G}_\Phi$, such that[1,2,4,5]

$$\rho = \int \omega\mu_\rho(d\omega) \tag{9}$$

(v) Two Gibbs states ρ and ρ' are equivalent (mutually absolutely continuous) if and only if μ_ρ and $\mu_{\rho'}$ are equivalent. This can be seen from the construction of μ_ρ in Refs. 4 and 5.

2. THE RESULTS

Definition. Two generalized b.c. b and b' have a *finite energy difference* if

$$C_{b,b'} = \sup_\Lambda \sup_{\Omega_\Lambda} \left| \sum_{X\subset\Lambda} (b_X{}^\Lambda(\sigma_X) - b_X'^\Lambda(\sigma_X)) \right| < \infty \tag{10}$$

For b.c. coming from configurations σ and σ' this means

$$\sup_\Lambda \sup_{\bar\sigma\in\Omega_\Lambda} \left| \sum_{\substack{X\cap\bar\Lambda\neq\varnothing\\ X\cap\Lambda\neq\varnothing}} [\Phi_X(\bar\sigma_X, \sigma_X) - \Phi_X(\bar\sigma_X, \sigma_X')] \right| < \infty \tag{11}$$

Theorem. Given two generalized b.c. b and b' having a finite energy difference and a sequence $\Lambda_n \nearrow \mathbb{Z}^d$ such that

$$\lim_{n\to\infty} \mu_{\Lambda_n,b} = \rho, \qquad \lim_{n\to\infty} \mu_{\Lambda_n,b'} = \rho'$$

then ρ and ρ' are equivalent and so are μ_ρ and $\mu_{\rho'}$ in (9). In particular, if ρ is extremal, $\rho = \rho'$.

Proof. Let $E_{\Lambda'}$ be a cylinder based on $\Omega_{\Lambda'}$, Λ' finite. We claim that, for any $\Lambda \supset \Lambda'$,

$$[\exp(-2\beta C_{b,b'})]\mu_{\Lambda,b}(E_{\Lambda'}) \leqslant \mu_{\Lambda,b'}(E_{\Lambda'}) \leqslant [\exp(2\beta C_{b,b'})]\mu_{\Lambda,b}(E_{\Lambda'}) \qquad (12)$$

The theorem follows then by letting $\Lambda = \Lambda_n$, $n \to \infty$, and using the fact that the monotone class, to which (12) extends immediately, generated by the cylinders is equal to the σ-algebra generated by them. To prove (12), we simply note that

$$\frac{Z_{\Lambda,b}}{Z_{\Lambda,b'}} = \frac{\int \exp[-\beta(H_{\Lambda,b}(\sigma_\Lambda) - H_{\Lambda,b'}(\sigma_\Lambda))] \exp[-\beta H_{\Lambda,b'}(\sigma_\Lambda)]\, \nu_\Lambda(d\sigma_\Lambda)}{\int \exp[-\beta H_{\Lambda,b'}(\sigma_\Lambda)]\, \nu_\Lambda(d\sigma_\Lambda)}$$
$$\leqslant \exp(\beta C_{b,b'})$$

by (10). The same inequality is true when the integrals in the numerator and the denominator are restricted to $E_{\Lambda'}$, which finishes the proof. ∎

Corollary 1. (See also Ref. 2, Chapter 5.) Let $d = 1$ and Φ be such that

$$\sum_{0\in X} \|\Phi_X\| \frac{(\operatorname{diam} X)}{|X|} < \infty \qquad (13)$$

then there is only one Gibbs state for Φ.

Proof. It is enough to show that any two b.c. have a finite energy difference, because then, by the theorem, all extremal Gibbs states (and therefore all possible Gibbs states) have to coincide.

To check (11), we note that $\forall \sigma, \sigma' \in \Omega$

$$\sum_{\substack{X\cap\Lambda\neq\varnothing\\ X\cap\bar\Lambda\neq\varnothing}} |\Phi_X(\tilde\sigma_\Lambda, \sigma_{\bar\Lambda}) - \Phi_X(\tilde\sigma_\Lambda, \sigma_{\bar\Lambda}')|$$
$$\leqslant 2 \sum_{\substack{X:\\ X\cap\Lambda\neq\varnothing\\ X\cup\bar\Lambda\neq\varnothing}} \|\Phi_X\|$$
$$= 2 \sum_{X:0\in X} \frac{\|\Phi_X\|}{|X|} (\#\{i\in\mathbb{Z} | (X+i)\cap\Lambda \neq \varnothing, (X+i)\cap\bar\Lambda \neq \varnothing\})$$

Since $d = 1$,

$$\#\{i\in\mathbb{Z} | (X+i)\cap\Lambda \neq \varnothing, (X+i)\cap\bar\Lambda \neq \varnothing\}$$
$$\leqslant 2\operatorname{diam}(X), \qquad \text{independent of } \Lambda \quad ∎$$

The next corollary deals with (the absence of) non-translation-invariant Gibbs states in two dimensions. We take a particular sequence of regions $B_{L,M} = \{i = (i_1, i_2) \in \mathbb{Z}^2 | \, |i_1| \leqslant M, \, |i_2| \leqslant L\}$ and we let first $M \to \infty$ and then $L \to \infty$. Since the first limit is in fact one-dimensional, the resulting state for fixed L in a band $B_L = \{i \in \mathbb{Z}^2 | \, |i_2| \leqslant L\}$ depends only on the b.c. on the sides of the band (this can be shown in the same way as for Corollary 1). It is known[6-8] in the two-dimensional, ferromagnetic, nearest neighbor Ising model [see (3)], that if we let $\sigma_i = +1$ for all i outside B_L with $i_1 \geqslant 0$ and $\sigma_i = -1$ for $i_1 < 0$, we have in the limit $L \to \infty$ a translation-invariant Gibbs state. We prove here a weaker result, which is valid, however, for more general systems, interactions, and b.c. For $\sigma \in \Omega$ we denote by $\mu_{B_L,\sigma}$ the unique Gibbs state in the band B_L with $\sigma_{\overline{B_L}}$ as b.c. Let $d_2(X) = \max\{\operatorname{dist}(i_2, j_2) | i, j \in X\}$ and $D_h{}^L = \{i \in \overline{B_L} | \, |i_1| \leqslant h\}$.

Corollary 2. Let Φ satisfy

$$\sum_{0 \in X} \|\Phi_X\| d_2(X) < \infty$$

Then, for any $\bar{\sigma}_1, \bar{\sigma}_2 \in \Omega_0$ and any $\sigma \in \Omega$ such that, for some $h \geqslant 0$,

$$\sigma_i = \bar{\sigma}_1, \qquad i_1 \geqslant h$$

$$\sigma_i = \bar{\sigma}_2, \qquad i_1 \leqslant -h$$

and any sequence (L_n), $L_n \to \infty$, such that $\lim_{n \to \infty} \mu_{B_{L_n},\sigma} = \rho$, then ρ and $T^{i_1}\rho$ are mutually absolutely continuous. If ρ is extremal, then it is translation invariant in the i_1 direction.

Proof. As in the preceding proof, we have to check (11) for σ and σ', where σ' is obtained by translating σ by one step in the i_1 direction [by definition (6), $\mu_{B_{L_n},\sigma'} \to T^{i_1}\rho$]. With these definitions of σ and σ', we have, for any rectangle $B_{L,M}$,

$$\Bigg| \sum_{\substack{X \cap B_{L,M} \neq \varnothing \\ X \cap \bar{B}_{L,M} \neq \varnothing}} [\Phi_X(\tilde{\sigma}_\Lambda, \sigma_{\bar{\Lambda}}) - \Phi_X(\tilde{\sigma}_\Lambda, \sigma_{\bar{\Lambda}}')] \Bigg|$$

$$\leqslant 2 \sum_{\substack{X \cap B_{L,M} \neq \varnothing \\ X \cap D_h^L \neq \varnothing}} \|\Phi_X\|$$

$$= 2 \sum_{0 \in X} \frac{\|\Phi_X\|}{|X|} (\#\{i \in \mathbb{Z}^2 | (X + i) \cap B_{L,M} \neq \varnothing \text{ and } (X + i) \cap D_L{}^h \neq \varnothing\})$$

and it is easily checked that this last number is bounded by $2(2h + 1)|X| d_2(X)$ uniformly in M and L. ∎

We consider now general ferromagnetic Ising spins and we show below that, in any dimension, some b.c. always lead to translation-invariant Gibbs states. Thus $\Omega_0 = \{-1, 1\}$, $\nu_0(-1) = \nu_0(+1) = 1/2$, and

$$\Phi_X = -J_X \prod_{i \in X} \sigma_i, \qquad J_X \geqslant 0 \tag{14}$$

The following is known about different b.c.:

Free b.c.: $b_X{}^\Lambda = 0$ in (7). Then $\lim_{\Lambda \nearrow \mathbb{Z}^d} \mu_{\Lambda,\text{free}} = \rho_0$ exists and is translation invariant.[9]

Plus (resp. minus b.c.): All $\sigma_i = +1$ (resp. -1) for $i \in \bar{\Lambda}$. Then $\lim_{\Lambda \nearrow \mathbb{Z}^d} \mu_{\Lambda,+} = \rho_+$ $(\lim_{\Lambda \nearrow \mathbb{Z}^d} \mu_{\Lambda,-} = \rho_-)$ exist and define translation-invariant Gibbs states, which are, moreover, extremal.[10]

Remark. Corollary 2 really says that if two b.c. coincide outside a "strip" (i.e. the set of sites between two parallel lines in $\mathbb{R}^2$) and the interactions are suitable in that strip, then these two b.c. lead to equivalent Gibbs states. Another simple but somewhat trivial application of this remark is the following: take a ferromagnetic system in two dimensions of the form (14) but with $J_X \neq 0$ only if X is the set of sites of an elementary square of the lattice. Then the symmetry group of this system[11] contains the flipping of all spins along any vertical or horizontal line of the lattice. Since the plus b.c. leads to an extremal Gibbs states ρ_+,[10] our observation shows that any b.c. obtained from the plus b.c. by flipping a finite number of lines coincide with ρ_+. But since this flipping is in fact a symmetry of the system, one concludes by a limiting argument that $\rho_+ = \rho_-$. One may easily construct similar examples. Actually this kind of model was studied in Ref. 12 under the name "trivial systems" and it is shown using the reduction procedure that they have a unique translation-invariant Gibbs state (and in fact a unique Gibbs state).

Now we shall restrict ourselves to the case where the set obtained from $E = \{X \mid J(X) > 0\}$ by the operation of symmetric difference applied to the elements of E contains all the even subsets of $\mathbb{Z}^d$. Then, the "symmetry group"[11] of the system is reduced to two elements, the identity and the flipping of all spins.

For all β, except possibly for a countable set,[13] and certainly for all β sufficiently large,[10,11] the state ρ_0 has the decomposition corresponding to (9):

$$\rho_0 = \tfrac{1}{2}(\rho_+ + \rho_-) \tag{15}$$

Corollary 3. For any generalized b.c. such that

$$\sup_\Lambda \sup_{\sigma_\Lambda \in \Omega_\Lambda} \Big| \sum_{X \subset \Lambda} b_X{}^\Lambda(\sigma_X) \Big| < \infty \tag{16}$$

and any sequence $\Lambda_n \to \mathbb{Z}^d$ such that $\lim_{n\to\infty} \mu_{\Lambda_n, b} = \rho$, ρ is translation invariant, whenever (15) holds.

Proof. This follows immediately from the theorem because any b.c. satisfying (16) and the free b.c. have a finite energy difference. So μ_ρ in (9) has to be absolutely continuous with respect to μ_{ρ_0}, which by (15) is concentrated on ρ_+ and ρ_-. ∎

[The interest of this corollary may be underlined by comparing it with point (i) in the discussion below.]

The last corollary concerns the semiinfinite, two-dimensional Ising model with nearest neighbor interaction. This is a model as in (3) but where our lattice $\mathbb{Z}^2$ is replaced by $\mathbb{L} = \{i = (i_1, i_2) \in \mathbb{Z}^2 | i_1 > 0\}$. We put some b.c. on the line $i_1 = 0$ and ask whether for a given b.c. there is a unique Gibbs state for the system in $\mathbb{L}$. This is known[7,8] when $\sigma_i = +1$ (or -1) for all i with $i_1 = 0$ (and is false for the free b.c. below the critical temperature found by Onsager). Here we extend this result to other b.c.

Corollary 4. In the above model, for any b.c. on the line $i_1 = 0$ with $\sigma_i = +1$ (or -1) for $|i_2| > N$ for some N, there is only one Gibbs state.

Proof. Take a sequence of regions in $\mathbb{L}$, $\Lambda_{L,M} = \{i \in L | i_1 \leqslant M, |i_2| \leqslant L\}$. If there is more than one Gibbs state for our system, then we can find [point (iii) in the Introduction] two configurations σ and σ' such that

$$\lim_{L,M\to\infty} \mu_{\Lambda_{L,M},\sigma} = \rho, \qquad \lim_{L,M\to\infty} \mu_{\Lambda_{L,M},\sigma'} = \rho'$$

are two different extremal, i.e., nonequivalent Gibbs states [point (v) in the Introduction]. But each of these b.c. will have a finite energy difference with the same b.c. for the system with $\sigma_i = +1$, $i_1 = 0$ (because we only put a finite number of -1 on the line $i_1 = 0$ in our system). Since there is a unique Gibbs state for the system $\sigma_i = +1$ on $i_1 = 0$, ρ and ρ' have to be equivalent to it and therefore equivalent among themselves, which contradicts the fact that they are different extremal Gibbs states. ∎

3. DISCUSSION

(i) We may ask whether the stability of extremal Gibbs states, expressed by the fact that, if a b.c. b yields an extremal ρ, then all b.c. b' differing from b by a finite energy also yield ρ, holds also for nonextremal Gibbs states. The answer is no. We can give several examples of nonextremal Gibbs states where it does not hold: if we take a ferromagnetic Ising model with nearest neighbor interactions and free boundary conditions, we know that it leads to a nonextremal Gibbs state for sufficiently large β.[10] (Presumably all β

above the critical β.) If we take $\Lambda_n \nearrow \mathbb{Z}^d$ and fix one spin adjacent to Λ_n, $\sigma_{j_n} = +1$, and otherwise free b.c., i.e., $b_X{}^\Lambda$ in (7) is $J_{ij_n}\sigma_i$ for $X = \{i\}$, we get a state with a strictly positive expectation value $\rho(\sigma_0) > 0$ for β sufficiently large. Obviously for the free b.c., $\rho_0(\sigma_0) = 0$, by symmetry. So changing only one spin on the boundary changes the Gibbs state. One proves that $\rho(\sigma_0) > 0$ with a Peierls argument in the form used in Ref. 14. If $\sigma_0 \neq 1$, there must be a contour (i.e., a connected set of bonds $\langle ij \rangle$ with $\sigma_i = \sigma_j$) that is crossing the line joining σ_0 and σ_{j_n} and such that 0 and j_n are separated by that contour. Since the probability of a contour containing k elements is less than $e^{-\beta Jk}$, the number of such contours containing a given element is bounded by 3^k and the cardinality of a contour separating 0 and j_n and crossing the line between 0 and j_n at a distance h from 0 and j_n has to be larger than $2h$, we can conclude the argument as usual to obtain $\langle \sigma_0 \rangle_{\Lambda_n, \sigma_{j_n}} = +1 \geqslant \delta > 0$ for β sufficiently large. In the two-dimensional model we expect the preceding result to hold in fact up to the critical temperature, because of the equivalence between the short- and the long-range order in that model.[15]

Since the free b.c. is not a true b.c., i.e., does not correspond to a configuration outside Λ, we may ask for an example with a true b.c. A simple answer comes from spin-1 systems, i.e., the spin takes the values -1, 0, $+1$. Then free b.c. are just the b.c. with $\sigma_j = 0$, $\forall j \in \bar{\Lambda}$, and the same Peierls argument works.

The interesting open question is whether the above stability completely characterizes extremal Gibbs states, i.e., does the requirement that ρ be obtainable from b.c. (true for extremal states) and be stable imply that ρ is extremal?

(ii) Our theorem applied to the above example tells us that ρ_0 and ρ are equivalent, so $\rho_0 \neq \rho$ implies that neither state is extremal. Interestingly enough, one can use a Peierls argument to show that in three dimensions the b.c. $\sigma_j = +1$, $j_1 = 0$, and free otherwise leads to the same (extremal) state as $\sigma_j = +1$ everywhere (state ρ_+) at low temperatures. First of all, this state ρ is translation invariant because by correlation inequalities

$$\langle \sigma_i \sigma_j \rangle_{\Lambda, b} \geqslant \langle \sigma_i \sigma_j \rangle_{\Lambda, \text{free b.c.}}$$

and this is enough by the results of Ref. 13 and the fact that at low temperatures there are only two extremal translation-invariant Gibbs states.[10] In these Gibbs states ρ_+ (resp. ρ_-) a Peierls argument shows that the set of configurations with an infinite cluster of minus spins (resp. plus spins) has measure zero. So, we have only to show that in ρ the probability that σ_0 belongs to an infinite cluster of -1 spins is zero. By translation invariance, this holds for any other point and a countable union of sets of measure zero is of measure zero. We use a slightly modified Peierls argument.

Suppose $\sigma_0 = -1$; then, due to the b.c., we can draw in the plane $i_1 = 0$

the smallest closed contour surrounding the origin. Now we consider the smallest connected contour S (in the same sense as above but on $\mathbb{Z}^3$) containing that contour in the plane. This new contour S may be open or closed. We define its interior as the set of points that can be connected to 0 through n.n. bonds that never intersect S and we estimate the probability of S by flipping all the spins in the interior of S. Therefore, this probability is bounded by $e^{-\beta|S|}$, $|S|$ is the number of n.n. bonds in S. But the number of contours with $|S| = n$ containing 0 in their interior is bounded by $3^n n^2$. So for $\beta > \log 3$ the probability that 0 is in the interior of an open contour, i.e., infinite contour, or, in other terms, belongs to an infinite cluster of -1 spins, is zero.

The following question is open: does this state coincide with ρ^+ at all temperatures? Or does it cease to be so at some temperature below the critical one? If this is the case, has that temperature anything to do with the presumed roughening temperature of this model[16] or the presumed percolation temperature[17]?

(iii) We have restricted our discussion to bounded (i.e., compact Ω_0) spin systems on a lattice and without hard cores. It is, however, possible to extend the theorem and its corollaries to continuous systems in $\mathbb{R}^d$, to some unbounded spins, and to some kind of hard core, keeping in mind the fact that we use only the boundedness of the ratio of the probability of certain events for the two boundary conditions. For systems with hard cores this would mean the following: two b.c. b and b' outside Λ define different admissible phase spaces $\Omega_\Lambda(b)$ and $\Omega_\Lambda(b')$ in Ω_Λ. Now, we would require that there exists for each Λ a $\Lambda' \subset \Lambda$ such that the restrictions to $\Omega_{\Lambda'}$ of $\Omega_\Lambda(b)$ and $\Omega_\Lambda(b')$ coincide and that

$$\sup_{\Lambda} \sup_{\sigma_{\Lambda\setminus\Lambda'}} |H_{\Lambda\setminus\Lambda'}(\sigma_{\Lambda\setminus\Lambda'})| < \infty$$

With this remark, one may extend Corollaries 1 and 2 to systems with hard cores. For example, in Ref. 2 one considers Ω_0 finite and the hard core is given by a matrix t, indexed by $\Omega_0 \times \Omega_0$, with $t_{\sigma\sigma'} = 0$ or 1 ($\sigma, \sigma' \in \Omega_0$), indicating whether nearest neighbor configurations are allowed or not. If there is an a such that t^a has all its entries strictly positive, then the system is called mixing. This means that the hard core is of finite range. Then, for $d = 1$ one may take as Λ' the set of points whose distance from $\bar{\Lambda}$ is larger than a. And, provided that the interaction satisfies (13), one proves that there is a unique Gibbs state for this system (as in Ref. 2) in the same way as Corollary 1. The same ideas extend to Corollary 2. One shows, e.g., that in both the continuum[18] and lattice[19] Widom–Rowlinson model the A–B b.c.[20] do not lead to an extremal nontranslation-invariant Gibbs state in two dimensions.

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